Kalb-Ramond interaction for a closed p-brane

Motowo Yamanobe[†], Sumio Ishikawa[‡], Yasuhiro Iwama,
Tadashi Miyazaki, Kazuyoshi Yamamoto and Ritsu Yoshida[*]
Department of Physics, Science University of Tokyo
Kagurazaka, Shinjuku-ku, Tokyo 162, Japan

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Abstract

The Kalb-Ramond action for an interacting string is generalized to the case of a high-dimensional object (p-brane). The interaction is found to be mediated by a gauge boson of a completely antisymmetric tensor of rank $p+1$.

[*]and Asia University Junior College
[†]E-mail: yamanobe@gradsun.ap.kagu.sut.ac.jp
[‡]E-mail: liszt@gradsun.ap.kagu.sut.ac.jp

The superstring theory was proposed as one of the candidates for the Theory of Everything (TOE). It had been obtaining brilliant success until the end of 1980s, when it was found to include some difficulties, such as those about physical dimensions and about degeneracy of vacuum states. Above all, the superstring theory tells nothing about the question why the extended object must be one-dimensional. Are there any possibility that a high-dimensional one be a candidate for the TOE? Along this line the attempt to extend an object of a one-dimensional string to a p-brane (p:an arbitrary positive integer) has been done. In order to recognize the p-brane theory with the aim to complete the TOE, we must elucidate it in all its aspects; for example, the classical solutions regulating the p-brane itself, its interactions and so on. In case of $p = 2$ (membrane), some investigations on its solutions expressing a free membrane have been done[3], but very little is known about the influence of interactions for the theory. The main purpose of the present paper is, therefore, to realize an interacting p-brane through the action-at-a-distance (AD) force.

Long time ago, Feynman and Wheeler showed that a charged particle interacting with an electromagnetic field can be described by the action-at-a-distance[1]. Kalb and Ramond extended their idea on a point particle to that on a one-dimensional extended object (string). They obtained a gauge boson as mediating the interaction between strings[2]. In case of a closed string, their gauge boson is an antisymmetric tensor of rank two and its degree of freedom is one. Namely, it becomes a massless scalar meson.

The main purpose of this paper is to generalize Kalb-Ramond's formulation to the the case of a *closed p-brane theory*. Let us consider an extended object of p dimensions (p-brane) which traces out D-dimensional space-time. Here $X_a^\mu(\xi_a^i)$ represents such a p-brane, where ξ_a^i $(i = 0 \sim p)$ are the parameters needed to represent the world sheet, and the subscript a labels each p-brane. The volume element of its world sheet is

$$d\sigma_a^{\mu_0\mu_1\cdots\mu_p} = d^{p+1}\xi_a \sigma_a^{\mu_0\mu_1\cdots\mu_p}, \tag{1}$$

where

$$\sigma_a^{\mu_0\mu_1\cdots\mu_p} = \frac{\partial\left(X_a^{\mu_0}, X_a^{\mu_1}, ..., X_a^{\mu_p}\right)}{\partial\left(\xi_a^0, \xi_a^1, ..., \xi_a^p\right)}. \tag{2}$$

The action for a free p-brane is given by

$$S_a^{free} = -\int_{\xi_a^0=\tau_i}^{\xi_a^0=\tau_f}\int_{\xi_a^1=0}^{\xi_a^1=a_1}\cdots\int_{\xi_a^p=0}^{\xi_a^p=a_p}(-d\sigma_a\cdot d\sigma_a)^{\frac{1}{2}}\,. \tag{3}$$

In case of a *closed* p-brane, with a boundary condition ($a_i > 0$)

$$X_a^\mu(\xi_a^0, 0, 0, ..., 0) = X_a^\mu(\xi_a^0, a_1, a_2, ..., a_p),$$

we obtain the equation of motion, by imposing that the action (3) be stationary under the variation

$$\begin{aligned} X_a^\mu(\xi_a^i) &\rightarrow X_a^\mu(\xi_a^i) + \delta X_a^\mu(\xi_a^i), \\ \delta X_a^\mu(\tau_i, \xi_a^1, ..., \xi_a^p) &= \delta X_a^\mu(\tau_f, \xi_a^1, ..., \xi_a^p) = 0, \end{aligned} \tag{4}$$

$$(p+1)D_a^{\mu_1\mu_2\ldots\mu_p}\left[\frac{\sigma_{a\ \mu\mu_1\mu_2\ldots\mu_p}}{(-\sigma_a\cdot\sigma_a)^{\frac{1}{2}}}\right] = 0, \tag{5}$$

where

$$\begin{aligned} D_a^{\mu_1\mu_2\ldots\mu_p} &= \sum_{i=0}^{p}\frac{\partial\sigma_a^{\mu\mu_1\ldots\mu_p}}{\partial X_{a\,,\,i}^\mu}\frac{\partial}{\partial\xi_a^i}, \\ X_{a\,,\,i}^{\mu_j} &= \frac{\partial X_a^{\mu_j}}{\partial\xi_a^i}. \end{aligned} \tag{6}$$

Now we take up two closed p-branes a and b interacting with each other through AD forces, and start with the following action

$$S = S_a^{free} + S_b^{free} + \int d^{p+1}\xi_a\int d^{p+1}\xi_b R_{ab}(X_a, X_b, X_{a\,,\,i}, X_{b\,,\,j}), \tag{7}$$

with

$$R_{ab} = R_{ba}, \tag{8}$$

for symmetry requirement. Imposing the condition that the action (7) be stationary under the variation (4), the equation of motion is obtained as follows:

$$(p+1)D_a^{\mu_1\mu_2\ldots\mu_p}\left[\frac{\sigma_{a\ \mu\mu_1\mu_2\ldots\mu_p}}{(-\sigma_a\cdot\sigma_a)^{\frac{1}{2}}}\right] = \int d^{p+1}\xi_b\left(\frac{\partial R_{ab}}{\partial X_a^\mu} - \frac{\partial}{\partial\xi_a^i}\frac{\partial R_{ab}}{\partial X_{a\,,\,i}^\mu}\right). \tag{9}$$

Multiplying Eq.(8) by $X^{\mu}_{a\,,\,i}$ $(i = 0 \sim p)$, we immediately have

$$0 = \left[\frac{\partial}{\partial \xi^i_a}\left(1 - \sigma_a^{\mu_0\mu_1\cdots\mu_p}\frac{\partial}{\partial \sigma_a^{\mu_0\mu_1\cdots\mu_p}}\right)\right]\int d^{p+1}\xi_b R_{ab}, \tag{10}$$

which shows that R_{ab} must be a linear function of the $\sigma_a^{\mu_0\mu_1\cdots\mu_p}$. Equation (10) can also be obtained by demanding that the action be invariant under the reparametrization

$$\xi^i_a \to \xi^i_a + \delta\xi^i_a. \tag{11}$$

Assuming that R_{ab} depends on $X^{\mu}_{a\,,\,i}$ only through the combinations of σ_a and σ_b, Eq. (9) is rewritten as

$$(p+1)D_a^{\mu_1\mu_2\ldots\mu_p}\left[\frac{\sigma_{a\ \mu\mu_1\mu_2\ldots\mu_p}}{(-\sigma_a\cdot\sigma_a)^{\frac{1}{2}}}\right] = \int d^{p+1}\xi_b\left(\frac{\partial R_{ab}}{\partial X^{\mu}_a} - (p+1)D_a^{\mu_1\mu_2\ldots\mu_p}\frac{\partial R_{ab}}{\partial \sigma_a^{\mu\mu_1\cdots\mu_p}}\right). \tag{12}$$

We explicitly take the following form as the interaction R_{ab},

$$R_{ab} = g_a g_b \sigma_a^{\mu_0\mu_1\cdots\mu_p}\sigma_{a\,\mu_0\mu_1\ldots\mu_p}G(S^2_{ab}), \tag{13}$$

with

$$S^2_{ab} = (X_a - X_b)\cdot(X_a - X_b),$$

where g_a and g_b are coupling constants and G is some Green's function in D-dimensional space-time, representing time-symmetric interactions, i.e., composed of two parts; advanced and retarded parts. In this case the equation of motion is reduced to the following,

$$(p+1)D_a^{\mu_1\mu_2\ldots\mu_p}\left[\frac{\sigma_{a\ \mu\mu_1\mu_2\ldots\mu_p}}{(-\sigma_a\cdot\sigma_a)^{\frac{1}{2}}}\right] = g_a F_b^{\mu\mu_0\cdots\mu_p}(X_a)\sigma_{a\,\mu_0\mu_1\ldots\mu_p}. \tag{14}$$

Note that Eq.(14) is much similar to the equation describing a charged particle under the Lorentz force. The analog of the electromagnetic field tensor is completely antisymmetric of rank $p+2$, and it can be written as

$$\begin{aligned} F_b^{\mu\mu_0\mu_1\cdots\mu_p}(X) \;\equiv\;& \partial^{\mu}\phi_b^{\mu_0\mu_1\cdots\mu_p}(X) \\ &+(-1)^{|p|\cdot 1}\partial^{\mu_0}\phi_b^{\mu_1\mu_2\cdots\mu}(X) \\ &+(-1)^{|p|\cdot 2}\partial^{\mu_1}\phi_b^{\mu_2\mu_3\cdots\mu_0}(X) \\ &\vdots \\ &+(-1)^{|p|\cdot(p+1)}\partial^{\mu_p}\phi_b^{\mu\mu_1\cdots\mu_{p-1}}(X), \end{aligned} \tag{15}$$

with

$$|p| = 0 \quad \text{for } p : \text{odd},$$

and

$$|p| = 1 \quad \text{for } p : \text{even},$$

where $\phi_b^{\mu_0\mu_1\cdots\mu_p}(X)$ is the tensor potential construct due to string b of rank $p+1$,

$$\phi_b^{\mu_0\mu_1\cdots\mu_p}(X) \quad = \quad g_b \int d^{p+1}\xi_b \sigma_b^{\mu_0\mu_1\cdots\mu_p} G((X - X_b)^2). \tag{16}$$

It is found, by an explicit calculation, that

$$\partial_{\mu_0} \phi_b^{\mu_0\mu_1\cdots\mu_p}(X) = 0. \tag{17}$$

$G(X)$ is the general Green's function in D dimensional space-time, and obeys the equation

$$\left(\partial^\mu \partial_\mu + m^2\right) G(X^2) = -C\delta^{(D)}(X), \tag{18}$$

where C is a dimensionless constant and m is to be identified with a mass of the field. By Eq. (18), we have

$$\left(\partial^\mu \partial_\mu + m^2\right) \phi_b^{\mu_0\mu_1\cdots\mu_p}(X) = -C j_b^{\mu_0\mu_1\cdots\mu_p}(X), \tag{19}$$

where $j_b^{\mu_0\mu_1\cdots\mu_p}(X)$ is the matter current,

$$j_b^{\mu_0\mu_1\cdots\mu_p}(X) = g_b \int d\sigma_b^{\mu_0\mu_1\cdots\mu_p} \delta^{(D)}(X - X_b). \tag{20}$$

These equations (18)-(20) express a gauge field mediating the interacting p-branes with one another.

On the analogy of the case of the string[2], we will transfer to the field theory. The gauge field only couples to the p-brane through the field tensor (15), and the coupling is invariant under the gauge transformation

$$\begin{aligned} \phi^{\mu_0\mu_1\cdots\mu_p} \longrightarrow \quad & \phi^{\mu_0\mu_1\cdots\mu_p} \\ & + \partial^{\mu_0} \Lambda^{\mu_1\mu_2\cdots\mu_p}(X) \end{aligned}$$

$$
\begin{aligned}
&+(-1)^{|p|\cdot 1}\partial^{\mu_1}\Lambda^{\mu_2\mu_3\cdots\mu_0}(X)\\
&+(-1)^{|p|\cdot 2}\partial^{\mu_2}\Lambda^{\mu_3\mu_4\cdots\mu_1}(X)\\
&\vdots\\
&+(-1)^{|p|\cdot p}\partial^{\mu_p}\Lambda^{\mu_0\mu_1\cdots\mu_{p-1}}(X),
\end{aligned}
\tag{21}
$$

where $\Lambda^{\mu_1\mu_2\cdots\mu_p}$ is completely antisymmetric with respect to its indices. The Lagrangian density invariant by the transformation (21), from which the equation of motion (14) is led, is given by

$$
\mathcal{L}(X) = \frac{(-1)^{|p|}}{2\cdot(p+2)!}F^{\mu\mu_0\mu_1\cdots\mu_p}(X)F_{\mu\mu_0\mu_1\cdots\mu_p}(X). \tag{22}
$$

The field $\phi^{\mu_0\mu_1\cdots\mu_p}$ is massless. The total action for the closed p-brane is, therefore,

$$
S = -\int\sqrt{-d\sigma\cdot d\sigma} + g\int d\sigma\cdot\phi + \int d^DX\mathcal{L}, \tag{23}
$$

with $\mathcal{L}$ given by Eq.(22). We immediately find that the degrees of freedom for the field $\phi^{\mu_0\mu_1\cdots\mu_p}$ are

$$
\begin{cases} {}_{D-2}C_{p+1} & \text{for } D \geq p+3, \\ 0 & \text{for } D = p+1,\ p+2. \end{cases} \tag{24}
$$

In conclusion, we have obtained the action for an interacting closed p-brane. The interaction term R_{ab} must be, and, in fact, *is* a linear function of the $\sigma_a^{\mu_0\mu_1\cdots\mu_p}$. One might have other interactions with the above linearity[4]. The equation of motion for the p-brane in our system is similar to that for a charged particle under the Lorentz force. By Eqs.(18)-(20) we see that the field $\phi^{\mu_0\mu_1\cdots\mu_p}$ is a gauge boson. It corresponds to the photon in the electromagnetic theory and its degrees of freedom are calculated to be ${}_{D-2}C_{p+1}$.

In this paper, we have confined ourselves to *closed* p-branes. The investigation on the interacting *open* p-branes will be reported shortly.

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